

Optimal Sample Complexity for Stable Matrix Recovery

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Abstract—Tremendous efforts have been made to study the theoretical and algorithmic aspects of sparse recovery and low-rank matrix recovery. This paper fills a theoretical gap in matrix recovery: the optimal sample complexity for stable recovery without constants or log factors. We treat sparsity, low-rankness, and potentially other parsimonious structures within the same framework: constraint sets that have small covering numbers or Minkowski dimensions. We consider three types of random measurement matrices (unstructured, rank-1, and symmetric rank-1 matrices), following probability distributions that satisfy some mild conditions. In all these cases, we prove a fundamental result – the recovery of matrices with parsimonious structures, using an optimal (or near optimal) number of measurements, is stable with high probability.

I. INTRODUCTION

Matrix recovery plays a central role in many applications of signal processing and machine learning. It is widely known that an unknown matrix can be recovered from an underdetermined system of linear measurements, by exploiting parsimonious structures of the matrix, such as sparsity or low-rankness [1], [2]. A special case where the unknown matrix is a sparse vector has been of particular interest in the context of compressed sensing and variable selection in linear regression.

Linear measurements of an unknown matrix are obtained through linear functionals, i.e., inner products with measurement matrices, which take different forms in different applications. In matrix completion [3], blind deconvolution via lifting [4], and bilinear regression [5], the measure matrices have rank-1. In phase retrieval via lifting [6], and in covariance matrix estimation via sketching [7], the measurement matrices are symmetric (or Hermitian) rank-1 matrices.

In practice, measurements are corrupted with additive noise. It is of interest to answer the question: under what conditions can the unknown matrix be estimated stably from noisy measurements. Many stability results are shown by demonstrating the effectiveness of convex relaxation. As for the recovery of sparse vectors, early results using the restricted isometry property (RIP) [8], [9] showed that stable recovery of s -sparse vectors of length n is guaranteed with $m = O(s \log(n/s))$ i.i.d. Gaussian random measurements. Later an RIPless analysis showed that, for a larger class of measurement functionals, $m = O(s \log(n/s))$ measurements are sufficient for stable recovery. The results on stable recovery of sparse vectors were extended to the case of low-rank matrices [10], guaranteeing

the recovery of $n \times n$ matrices of rank- r from $m = O(rn \log n)$ linear measurements. Candès and Plan [11] sharpened sample complexity to $m = O(rn)$. Chandrasekaran et al. unified the parsimonious models including low-rank matrices and sparse vectors as atomic sparsity models [12]. Using the Gaussian width of a tangent cone, they computed sample complexities for stable recovery that coincide with the empirical phase transition using convex relaxation. Recently, recovery of matrices that are sparse and low-rank is studied (e.g., [13]). As for rank-1 measurement matrices, Cai and Zhang [14] showed that stable recovery of $n_1 \times n_2$ matrices of rank r is achieved by $m = O(r(n_1 + n_2))$ measurements. Recently, stable recovery in blind deconvolution and phase retrieval [15], [6], [16] has been studied by lifting to matrix recovery.

Another line of work studies the information-theoretic fundamental limit of sparse or low-rank matrix recovery, establishing the sample complexities achieved by an optimal decoder (practical or not). Wu and Verdù studied the performance of the optimal stable decoder for compressed sensing in a Bayesian framework [17]. Also for compressed sensing, Reeves showed, without a prior distribution on the unknown sparse vector, the optimal sample complexity for stable recovery from i.i.d. Gaussian random measurements [18]. Riegler et al. studied the information-theoretic limit for the unique recovery of matrices in a set of small Minkowski dimension, using unstructured or rank-1 measurement matrices [19]. However, a relevant result on stable matrix recovery has been missing.

In this paper, we address the fundamental question of stable matrix recovery: how many measurements are sufficient to guarantee the existence of stable decoder? Similar to the paper by Riegler et al. [19], our analysis covers a large category of problems, including compressed sensing, low-rank matrix recovery, phase retrieval, etc.

II. PROBLEM STATEMENT

A. Notations

The unit ball with respect to the ℓ_2 norm in \mathbb{R}^n (resp. with respect to the Frobenius norm in $\mathbb{R}^{n_1 \times n_2}$) centered at the origin is denoted by \mathcal{B}_n (resp. $\mathcal{B}_{n_1 \times n_2}$). We use V_n to denote the volume of a unit ball in \mathbb{R}^n , and $\eta_n = V_{n-1}/V_n$.

B. Matrix Recovery

We study constrained matrix recovery. Suppose X_0 is an unknown $n_1 \times n_2$ matrix. We have m linear measurements,

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$y = \mathcal{A}(X_0) + e \in \mathbb{R}^m$, where the i -th component of $\mathcal{A}(X_0)$ is in the form $\langle A_i, X_0 \rangle$ for $i \in [m]$, $\{A_i\}_{i=1}^m \subset \mathbb{R}^{n_1 \times n_2}$ denote the measurement matrices, and $e \in \mathbb{R}^m$ denotes the noise or other distortions in the measurement. The matrix recovery problem refers to estimating the unknown matrix X_0 from y . We consider three models for the measurement matrices:

- 1) *Unstructured* measurement matrices $\{A_j\}_{j=1}^m$.
- 2) *Rank-1* measurement matrices $\{a_j b_j^T\}_{j=1}^m$.
- 3) *Symmetric rank-1* measurement matrices $\{a_j a_j^T\}_{j=1}^m$.

For the unstructured case, $\{A_j\}_{j=1}^m$ are i.i.d. random matrices following one of the two probability distributions: (1) uniform distribution on $\mathcal{RB}_{n_1 \times n_2}$; or (2) Gaussian distribution $N(0, \sigma^2 I_{n_1 n_2})$. For the rank-1 or symmetric rank-1 cases, $\{a_j\}_{j=1}^m$ (resp. $\{b_j\}_{j=1}^m$) are i.i.d. random vectors following: (1) uniform distribution on a ball in \mathbb{R}^{n_1} (resp. \mathbb{R}^{n_2}); or (2) Gaussian distribution with i.i.d. entries.

In matrix recovery, the number of measurements m is often smaller than $n_1 n_2$ – the number of entries in X_0 . For matrix recovery to be well-posed, the unknown matrix X_0 is assumed to belong to a known constraint set $\Omega_{\mathcal{X}} \subset \mathbb{R}^{n_1 \times n_2}$, which encodes our prior knowledge of X_0 . As examples, we consider the following constraint sets:

(I) *Matrices in a subspace*: a subspace of $\mathbb{R}^{n_1 \times n_2}$, of dimension t . Examples of such subspaces include the sets of Hankel matrices, Toeplitz matrices, and symmetric matrices.

(II) *Sparse matrices*: the set of $n_1 \times n_2$ matrices s -sparse over a dictionary, whose atoms are M_1, M_2, \dots, M_t . (When $n_2 = 1$, the sparse matrix recovery problem reduces to sparse vector recovery.) Let $\mathbf{M} = [\text{vec}(M_1), \text{vec}(M_2), \dots, \text{vec}(M_t)]$, then $\text{vec}(X) = \mathbf{M}\beta$. The sparsity-restricted condition number is defined by

$$\kappa_s = \frac{\max_{\|\beta\|_2=1, \|\beta\|_0 \leq s} \|\mathbf{M}\beta\|_2}{\min_{\|\beta\|_2=1, \|\beta\|_0 \leq s} \|\mathbf{M}\beta\|_2}.$$

For example, if \mathbf{M} is an orthonormal basis (e.g., the standard basis), then $\kappa_s = 1$. If \mathbf{M} has a restricted isometry constant δ_s [8], then $\kappa_s \leq \sqrt{(1+\delta_s)/(1-\delta_s)}$.

(III) *Low-rank matrices*: the set of $n_1 \times n_2$ matrices of rank at most r .

(IV) *Sparse low-rank matrices*: the set of $n_1 \times n_2$ matrices that have rank at most r , have at most s_1 nonzero rows, and have at most s_2 nonzero columns ($r < \min\{s_1, s_2\}$).

(V) *Symmetric low-rank matrices*: the set of $n \times n$ symmetric matrices of rank at most r .

(VI) *Symmetric sparse low-rank matrices*: the set of $n \times n$ symmetric matrices that have rank at most r , and have at most s nonzero rows (or columns).

Note that all the above constraint sets are cones. For all practical purposes, the matrix X_0 has finite energy. In this paper, we consider the constraint set restricted to the unit ball:

$$\Omega_{\mathcal{B}} := \Omega_{\mathcal{X}} \cap \mathcal{B}_{n_1 \times n_2}.$$

Then we can estimate X_0 , for example, by solving the following constrained least squares problem:

$$\begin{aligned} \text{(LSMR)} \quad & \min_X \quad \|\mathcal{A}(X) - y\|_2, \\ & \text{s.t. } X \in \Omega_{\mathcal{B}}. \end{aligned}$$

C. Stability

Definition 1. We say that the recovery of $X_0 \in \Omega_{\mathcal{B}}$ using measurement operator \mathcal{A} is stable at level (δ, ε) , if for all $X \in \Omega_{\mathcal{B}}$ such that $\|\mathcal{A}(X) - \mathcal{A}(X_0)\|_2 \leq \delta$, we have $\|X - X_0\|_{\mathcal{X}} \leq \varepsilon$. Here, $\|\cdot\|_{\mathcal{X}}$ can either be the Frobenius norm $\|\cdot\|_F$ or the spectral norm $\|\cdot\|_2$, and $\varepsilon = \varepsilon(\delta)$ is a function of δ that vanishes as δ approaches 0.

If the recovery of X_0 is stable, then for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $X \in \Omega_{\mathcal{B}}$ that satisfy $\|\mathcal{A}(X) - \mathcal{A}(X_0)\|_2 \leq \delta$, we have $\|X - X_0\|_{\mathcal{X}} \leq \varepsilon$. If \mathcal{A} (restricted to the domain $\Omega_{\mathcal{B}}$) is invertible, i.e., there exists $\mathcal{A}^{-1} : \mathcal{A}(\Omega_{\mathcal{B}}) \rightarrow \Omega_{\mathcal{B}}$, then stability at X_0 implies that \mathcal{A}^{-1} is continuous at $\mathcal{A}(X_0)$.

Stability, as defined above, guarantees the accuracy of the constrained least squares estimation. Let X_1 denote the solution to (LSMR). Suppose the perturbation in the measurement is small, $\|e\|_2 \leq \frac{\delta}{2}$ for some small $\delta > 0$. Then the deviation of $\mathcal{A}(X_1)$ from $\mathcal{A}(X_0)$ is small, i.e.,

$$\begin{aligned} \|\mathcal{A}(X_1) - \mathcal{A}(X_0)\|_2 &\leq \|\mathcal{A}(X_1) - y\|_2 + \|\mathcal{A}(X_0) - y\|_2 \\ &\leq 2\|\mathcal{A}(X_0) - y\|_2 = 2\|e\|_2 \leq \delta. \end{aligned}$$

By the definition of stability, we have $\|X_1 - X_0\|_{\mathcal{X}} \leq \varepsilon$, which is also a small quantity.

Suppose $\Omega_{\mathcal{X}}$ is a cone, and we need to evaluate the stability on a bounded constraint set $L\Omega_{\mathcal{B}}$ ($L < \infty$). We can scale X_0 and the radius of the ball by $\frac{1}{L}$ simultaneously. If for all $X \in \Omega_{\mathcal{B}}$ such that $\|\mathcal{A}(X) - \mathcal{A}(\frac{X_0}{L})\|_2 \leq \delta$ we have $\|X - \frac{X_0}{L}\|_{\mathcal{X}} \leq \varepsilon(\delta)$, then for all $X \in L\Omega_{\mathcal{B}}$ such that $\|\mathcal{A}(X) - \mathcal{A}(X_0)\|_2 \leq \delta$, we have $\|X - X_0\|_{\mathcal{X}} \leq L\varepsilon(\frac{\delta}{L})$. Hence stability on $\Omega_{\mathcal{B}}$ implies stability on any bounded subset of $\Omega_{\mathcal{X}}$. Therefore, without loss of generality, we consider $\Omega_{\mathcal{B}}$ as a representative for bounded constraint sets.

III. MAIN RESULTS

A. Unstructured Measurement Matrices

Theorem 1. Suppose $\Omega_{\mathcal{X}}$ is one of the sets in (I) – (IV), and the measurement matrices $\{A_j\}_{j=1}^m$ are i.i.d. random matrices following distribution \mathcal{D} . If $m > d$, and $\delta < R$, then the recovery of an arbitrary $X_0 \in \Omega_{\mathcal{B}}$ is stable at level (δ, ε) with probability $1 - P_f$.

1) If \mathcal{D} is the uniform distribution on $\mathcal{RB}_{n_1 \times n_2}$, then

$$P_f \leq C (6\eta_{n_1 n_2})^m (\delta/R)^{m-d} (1/\varepsilon)^m.$$

2) If \mathcal{D} is the Gaussian distribution $N(0, \sigma^2 I_{n_1 n_2})$, then

$$\begin{aligned} P_f &\leq \left(\frac{3\sqrt{2}}{\sqrt{\pi}\sigma} \right)^m R^d \cdot \frac{\delta^{m-d}}{\varepsilon^m} \\ &\quad + m \cdot e^{-\frac{n_1 n_2}{2} \left(\frac{R^2}{n_1 n_2 \sigma^2} - 1 - \ln \frac{R^2}{n_1 n_2 \sigma^2} \right)}, \quad \forall R > \sqrt{n_1 n_2} \sigma. \end{aligned}$$

TABLE I
A SUMMARY OF THE CONSTANTS

$\Omega_{\mathcal{X}}$	d	C
(I)	t	3^t
(II)	s	$(3\kappa_{2s})^s \cdot \binom{t}{s}$
(III)	$(n_1 + n_2)r$	$(6\sqrt{r})^{(n_1 + n_2)r}$
(IV)	$(s_1 + s_2)r$	$(6\sqrt{r})^{(s_1 + s_2)r} \cdot \binom{n_1}{s_1} \binom{n_2}{s_2}$
(V)	nr	$(r + 1) (6\sqrt{r})^{nr}$
(VI)	sr	$(r + 1) (6\sqrt{r})^{sr} \cdot \binom{n}{s}$

Here, the constants C, d depend on $\Omega_{\mathcal{B}} = \Omega_{\mathcal{X}} \cap \mathcal{B}_{n_1 \times n_2}$ (see Table I), and recovery error is measured in Frobenius norm.

We interpret the result for the uniform distribution case under two scenarios. The Gaussian distribution case can be interpreted similarly by choosing a proper expression for R .

1) *Noiseless measurement*: Let $\varepsilon = 6\eta_{n_1 n_2} C^{\frac{1}{m}} \left(\frac{\delta}{R}\right)^{\frac{m-d}{2m}}$, then $P_f \leq \left(\frac{\delta}{R}\right)^{(m-d)/2}$. When the perturbation δ approaches 0, both the error bound ε and the probability of failure P_f vanish. It follows that, if the measurements are without perturbation, then matrix recovery is unique almost surely under the same sample complexity $m > d$.

2) *Noisy measurement*: For the constrained least squares estimator X_1 , if the perturbation has norm $\|e\|_2 = \delta/2$, then $\mathbb{P}[\|X_1 - X_0\|_F > \varepsilon] \leq P_f$. Let $\varepsilon_0 = 6\eta_{n_1 n_2} C^{\frac{1}{m}} \left(\frac{\delta}{R}\right)^{\frac{m-d}{m}}$. Then $P_f(\varepsilon) \leq \left(\frac{\varepsilon_0}{\varepsilon}\right)^m$. Therefore,

$$\begin{aligned} \mathbb{E}[\|X_1 - X_0\|_F^2] &= \int_0^\infty \mathbb{P}[\|X_1 - X_0\|_F^2 > \varepsilon^2] d\varepsilon^2 \\ &\leq \int_0^{\varepsilon_0} 2\varepsilon \cdot P_f(\varepsilon) d\varepsilon + \int_{\varepsilon_0}^\infty 2\varepsilon \cdot P_f(\varepsilon) d\varepsilon^2 \\ &\leq \int_0^{\varepsilon_0} 2\varepsilon d\varepsilon + \int_{\varepsilon_0}^\infty 2\varepsilon \cdot \left(\frac{\varepsilon_0}{\varepsilon}\right)^m d\varepsilon \\ &= \varepsilon_0^2 + \frac{2\varepsilon_0^2}{m-2} \\ &= \frac{36m \cdot \eta_{n_1 n_2}^2 C^{\frac{2}{m}}}{m-2} \left(\frac{4\|e\|_2^2}{R^2}\right)^{\frac{m-d}{m}}. \end{aligned}$$

B. Rank-1 Measurement Matrices

In this section, we show that the same sample complexities as in Section III-A apply to matrix recovery with rank-1 measurement matrices of the form $\{A_j = a_j b_j^T\}_{j=1}^m$.

Theorem 2. Suppose $\Omega_{\mathcal{X}}$ is one of the sets in (I) – (IV), and the measurement matrices $\{A_j = a_j b_j^T\}_{j=1}^m$ satisfy that $\{a_j\}_{j=1}^m$ and $\{b_j\}_{j=1}^m$ are independent random vectors, where $\{a_j\}_{j=1}^m$ (resp. $\{b_j\}_{j=1}^m$) are i.i.d. following \mathcal{D}_1 (resp. \mathcal{D}_2). If $m > d$, and $\delta < R_1 R_2$, then the recovery of an arbitrary $X_0 \in \Omega_{\mathcal{B}}$ is stable at level (δ, ε) with probability $1 - P_f$.

1) If \mathcal{D}_1 and \mathcal{D}_2 are uniform distributions on $R_1 \mathcal{B}_{n_1}$ and $R_2 \mathcal{B}_{n_2}$, respectively, then

$$P_f \leq C \left(12\eta_{n_1} \eta_{n_2} \left(1 + \ln \frac{2R_1 R_2}{3\delta} \right) \right)^m \cdot \left(\frac{\delta}{R_1 R_2} \right)^{m-d} \left(\frac{1}{\varepsilon} \right)^m.$$

2) If \mathcal{D}_1 and \mathcal{D}_2 are Gaussian distributions $N(0, \sigma_1^2 I_{n_1})$ and $N(0, \sigma_2^2 I_{n_2})$, respectively, then

$$P_f \leq C \left(\frac{3}{\sigma_1 \sigma_2} \left(1 + \ln \left(1 + \frac{2\sigma_1 \sigma_2}{3\delta} \right) \right) \right)^m (R_1 R_2)^d \cdot \frac{\delta^{m-d}}{\varepsilon^m} + m e^{-\frac{n_1}{2} \left(\frac{R_1^2}{n_1 \sigma_1^2} - 1 - \ln \frac{R_1^2}{n_1 \sigma_1^2} \right)} + m e^{-\frac{n_2}{2} \left(\frac{R_2^2}{n_2 \sigma_2^2} - 1 - \ln \frac{R_2^2}{n_2 \sigma_2^2} \right)},$$

$$\forall R_1 > \sqrt{n_1} \sigma_1, \forall R_2 > \sqrt{n_2} \sigma_2.$$

Here, the constants C, d depend on $\Omega_{\mathcal{B}} = \Omega_{\mathcal{X}} \cap \mathcal{B}_{n_1 \times n_2}$ (see Table I), and recovery error is measured in spectral norm.

C. Symmetric Rank-1 Measurement Matrices

Theorem 3. Suppose $\Omega_{\mathcal{X}}$ is one of the sets in (I), (II), (V), and (IV), and all matrices in $\Omega_{\mathcal{X}}$ are symmetric. Suppose the measurement matrices $\{A_j = a_j a_j^T\}_{j=1}^m$ satisfy that $\{a_j\}_{j=1}^m$ are i.i.d. random vectors following a distribution \mathcal{D} . If $m > 2d$, and $\delta < R^2$, then the recovery of an arbitrary $X_0 \in \Omega_{\mathcal{B}}$ is stable at level (δ, ε) with probability $1 - P_f$.

1) If \mathcal{D} is the uniform distribution on $R \mathcal{B}_n$, then

$$P_f \leq C \left(2\sqrt{6} \cdot \eta_n \right)^m \left(\frac{\delta}{R^2} \right)^{m/2-d} \left(\frac{1}{\varepsilon} \right)^{m/2}.$$

2) If \mathcal{D} is the Gaussian distribution $N(0, \sigma^2 I_n)$, then

$$P_f \leq C \left(\frac{2\sqrt{3}}{\sqrt{\pi}\sigma} \right)^m \cdot R^{2d} \cdot \frac{\delta^{m/2-d}}{\varepsilon^{m/2}} + m e^{-\frac{n}{2} \left(\frac{R^2}{n\sigma^2} - 1 - \ln \frac{R^2}{n\sigma^2} \right)},$$

$$\forall R > \sqrt{n}\sigma.$$

Here, the constants C, d depend on $\Omega_{\mathcal{B}} = \Omega_{\mathcal{X}} \cap \mathcal{B}_{n_1 \times n_2}$ (see Table I), and recovery error is measured in spectral norm.

In phase retrieval, the measurements of an unknown vector $x_0 \in \mathbb{R}^n$ are obtained without signs. By Theorem 3, in the lifted phase retrieval problem, we need $m > 2d = 2n$ measurements to recover the unknown $n \times n$ symmetric rank-1 matrix $X_0 = x_0 x_0^T$. By Theorem 1, if the measurements are obtained with signs, $m > d = n$ measurements are sufficient. Hence, due to the loss of signs, we need twice as many measurements to recover the unknown vector stably.

IV. PROOF OF MAIN RESULTS

A. Covering Number

The constraint sets in Section II-B have a small description complexity quantified in terms of the covering number of $\Omega_{\mathcal{B}}$.

Definition 2. The covering number of a nonempty bounded set $\Omega_{\mathcal{B}} \subset \mathbb{R}^{n_1 \times n_2}$ is defined by:

$$N_{\Omega_{\mathcal{B}}}(\rho) := \min \left\{ N \in \mathbb{Z}^+ : \exists X_i \in \mathbb{R}^{n_1 \times n_2}, i = 1, 2, \dots, N \right. \\ \left. \text{s.t. } \Omega_{\mathcal{B}} \subset \bigcup_{i \in \{1, 2, \dots, N\}} (X_i + \rho \mathcal{B}_{n_1 \times n_2}) \right\}.$$

Lemma 1. If $\Omega_{\mathcal{X}}$ is one of the sets in (I) – (VI), then the covering number of $\Omega_{\mathcal{B}} = \Omega_{\mathcal{X}} \cap \mathcal{B}_{n_1 \times n_2}$ satisfies $N_{\Omega_{\mathcal{B}}}(\rho) \leq C \rho^{-d}$ for all $0 < \rho < 1$, where d and C are constants defined in Table I.

As shown in Section IV-C, the stability results rely on Ω_B only through the upper bound on the covering number in the form of $N_{\Omega_B}(\rho) \leq C\rho^{-d}$. Hence the results in this paper can be easily generalized to other constraint sets that admit similar upper bounds on the covering number.

B. Concentration of Measure

In Section III, we presented the results for random measurement matrices following uniform or Gaussian distributions. In fact, similar results can be derived for a large category of probability distributions, which satisfy the concentration of measure bounds in this section.

1) *Unstructured Measurement Matrices:* We assume that $\{A_j\}_{j=1}^m \subset \mathbb{R}^{n_1 \times n_2}$ are i.i.d. random matrices following a distribution \mathcal{D} that satisfies the following concentration of measure bounds:

$$\mathbb{P}_{\mathcal{D}}[\|A\|_F > R] \leq \theta_{\mathcal{D},R}, \quad (1)$$

$$\mathbb{P}_{\mathcal{D}}[|\langle A, X \rangle| \leq \delta] \leq C_{\mathcal{D}} \cdot \frac{\delta}{\varepsilon}, \quad \forall X \text{ s.t. } \|X\|_F \geq \varepsilon. \quad (2)$$

We have the following bounds for uniform distribution and Gaussian distribution. We omit the proof due to space limitations.

Lemma 2. For \mathcal{U} – uniform distribution on $RB_{n_1 \times n_2}$,

$$\theta_{\mathcal{U},R} = 0, \quad C_{\mathcal{U}} = \frac{2\eta_{n_1 n_2}}{R}.$$

Lemma 3. For \mathcal{G} – Gaussian distribution $N(0, \sigma^2 I_{n_1 n_2})$,

$$\theta_{\mathcal{G},R} = e^{-\frac{n_1 n_2}{2} \left(\frac{R^2}{n_1 n_2 \sigma^2} - 1 - \ln \frac{R^2}{n_1 n_2 \sigma^2} \right)}, \quad \forall R > \sqrt{n_1 n_2} \sigma,$$

$$C_{\mathcal{G}} = \frac{\sqrt{2}}{\sqrt{\pi} \sigma}.$$

2) *Rank-1 Measurement Matrices:* We assume that $\{A_j = a_j b_j^T\}_{j=1}^m$ satisfy that $\{a_j\}_{j=1}^m$ and $\{b_j\}_{j=1}^m$ are independent random vectors, where $\{a_j\}_{j=1}^m$ (resp. $\{b_j\}_{j=1}^m$) are i.i.d. following a distribution \mathcal{D}_1 (resp. \mathcal{D}_2), and $\mathcal{D}_1, \mathcal{D}_2$ satisfy the following concentration of measure bounds:

$$\mathbb{P}_{\mathcal{D}_1}[\|a\|_2 > R_1] \leq \theta_{\mathcal{D}_1, R_1}, \quad \mathbb{P}_{\mathcal{D}_2}[\|b\|_2 > R_2] \leq \theta_{\mathcal{D}_2, R_2},$$

$$\mathbb{P}_{\mathcal{D}_1 \mathcal{D}_2}[|a^T X b| \leq \delta] \leq C_{\mathcal{D}_1, \mathcal{D}_2, \delta} \cdot \frac{\delta}{\varepsilon}, \quad \forall X \text{ s.t. } \varepsilon \leq \|X\|_2 \leq 2.$$

Lemma 4. For \mathcal{U}_1 and \mathcal{U}_2 – uniform distributions on $R_1 \mathcal{B}_{n_1}$ and $R_2 \mathcal{B}_{n_2}$,

$$\theta_{\mathcal{U}_1, R_1} = \theta_{\mathcal{U}_2, R_2} = 0,$$

$$C_{\mathcal{U}_1, \mathcal{U}_2, \delta} = \frac{4\eta_{n_1} \eta_{n_2}}{R_1 R_2} \left(1 + \ln \frac{2R_1 R_2}{\delta} \right).$$

Lemma 5. For \mathcal{G}_1 and \mathcal{G}_2 – Gaussian distributions $N(0, \sigma_1^2 I_{n_1})$ and $N(0, \sigma_2^2 I_{n_2})$,

$$\theta_{\mathcal{G}_i, R_i} = e^{-\frac{n_i}{2} \left(\frac{R_i^2}{n_i \sigma_i^2} - 1 - \ln \frac{R_i^2}{n_i \sigma_i^2} \right)}, \quad \forall R_i > \sqrt{n_i} \sigma_i, \quad i = 1, 2,$$

$$C_{\mathcal{G}_1, \mathcal{G}_2, \delta} = \frac{1}{\sigma_1 \sigma_2} \left(1 + \ln \left(1 + \frac{2\sigma_1 \sigma_2}{\delta} \right) \right).$$

3) *Symmetric Rank-1 Measurement Matrices:* We assume that $\{A_j = a_j a_j^T\}_{j=1}^m$ satisfy that $\{a_j\}_{j=1}^m$ are i.i.d. random vectors following a distribution \mathcal{D} that satisfies the following concentration of measure bounds:

$$\mathbb{P}_{\mathcal{D}}[\|a\|_2 > R] = \theta_{\mathcal{D},R},$$

$$\mathbb{P}_{\mathcal{D}}[|a^T X a| \leq \delta] \leq C_{\mathcal{D}} \cdot \sqrt{\frac{\delta}{\varepsilon}}, \quad \forall X \text{ s.t. } \|X\|_2 \geq \varepsilon.$$

Lemma 6. For \mathcal{U} – uniform distributions on RB_n ,

$$\theta_{\mathcal{U},R} = 0, \quad C_{\mathcal{U}} = \frac{2\sqrt{2}\eta_n}{R}.$$

Lemma 7. For \mathcal{G} – Gaussian distributions $N(0, \sigma^2 I_n)$,

$$\theta_{\mathcal{G},R} = e^{-\frac{n}{2} \left(\frac{R^2}{n \sigma^2} - 1 - \ln \frac{R^2}{n \sigma^2} \right)}, \quad \forall R > \sqrt{n} \sigma,$$

$$C_{\mathcal{G}} = \frac{2}{\sqrt{\pi} \sigma}.$$

C. Proof Sketch of the Main Results

Due to space limitations and the similarity between proofs for unstructured, rank-1, and symmetric rank-1 measurement matrices, we provide only the proof for the unstructured case.

We present the following key lemma (Lemma 8), in terms of covering number of the constraint set, and the concentration of measure bounds for the random measurement matrices. All the results in Theorem 1 can be deduced from Lemmas 1, 2, 3, and 8.

Lemma 8. Suppose the covering number of Ω_B satisfies $N_{\Omega_B}(\rho) \leq C\rho^{-d}$ for all $0 < \rho < 1$, and the measurement matrices $\{A_j\}_{j=1}^m \subset \mathbb{R}^{n_1 \times n_2}$ are i.i.d. random matrices following a distribution \mathcal{D} that satisfies concentration of measure bounds in (1) and (2). If $m > d$, and $\delta < R$, then the stability results hold with probability $1 - P_f$, where

$$P_f \leq C (3C_{\mathcal{D}})^m \cdot R^d \cdot \frac{\delta^{m-d}}{\varepsilon^m} + m \cdot \theta_{\mathcal{D},R}.$$

Proof: The probability of failure for single point stability is:

$$P_f = \mathbb{P}_{\mathcal{D}^m}[\exists X \in \Omega_B - X_0, \text{ s.t. } \|X\|_F > \varepsilon \text{ and } \|\mathcal{A}(X)\|_2 \leq \delta].$$

Define $\Omega_{\varepsilon} := \{X \in \Omega_B - X_0 : \|X\|_F > \varepsilon\}$. Then

$$\begin{aligned} P_f &= \mathbb{P}_{\mathcal{D}^m}[\exists X \in \Omega_{\varepsilon} \text{ s.t. } \|\mathcal{A}(X)\|_2 \leq \delta] \\ &\leq \mathbb{P}_{\mathcal{D}^m} \left[\max_{j \in [m]} \|A_j\|_F \leq R, \text{ \& } \exists X \in \Omega_{\varepsilon} \text{ s.t. } \|\mathcal{A}(X)\|_2 \leq \delta \right] \\ &\quad + \mathbb{P}_{\mathcal{D}^m} [\exists j \in [m] \text{ s.t. } \|A_j\|_F > R] \\ &\leq \mathbb{P}_{\mathcal{D}^m} \left[\max_{j \in [m]} \|A_j\|_F \leq R, \text{ \& } \exists X \in \Omega_{\varepsilon} \text{ s.t. } \|\mathcal{A}(X)\|_2 \leq \delta \right] \\ &\quad + m \cdot \theta_{\mathcal{D},R}, \end{aligned} \quad (3)$$

where (3) follows from (1) and a union bound. To complete the proof, we need to bound the first term.

We form a minimal cover of Ω_{ε} with balls of radius $\rho = \frac{\delta}{R} < 1$ centered at the points $\{X_i\}_{i=1}^{N_{\Omega_{\varepsilon}}(\rho)}$. The centers of the balls are not necessarily in Ω_{ε} . However, by the minimality of

the cover, the intersection of Ω_ε with each ball is nonempty, hence there exists another set of points $\{X'_i\}_{i=1}^{N_{\Omega_\varepsilon}(\rho)}$ such that

$$X'_i \in \Omega_\varepsilon \cap (X_i + \rho\mathcal{B}_{n_1 \times n_2}), \quad i = 1, 2, \dots, N_{\Omega_\varepsilon}(\rho).$$

Now we can cover Ω_ε with balls of radius 2ρ centered at $\{X'_i\}_{i=1}^{N_{\Omega_\varepsilon}(\rho)}$, which are points in Ω_ε (a property that will be needed for inequality (8) below), because

$$(X_i + \rho\mathcal{B}_{n_1 \times n_2}) \subset (X'_i + 2\rho\mathcal{B}_{n_1 \times n_2}), \quad i = 1, 2, \dots, N_{\Omega_\varepsilon}(\rho),$$

$$\Omega_\varepsilon \subset \bigcup_{i \in [N_{\Omega_\varepsilon}(\rho)]} (X_i + \rho\mathcal{B}_{n_1 \times n_2}) \subset \bigcup_{i \in [N_{\Omega_\varepsilon}(\rho)]} (X'_i + 2\rho\mathcal{B}_{n_1 \times n_2}).$$

Therefore, the first term in (3) satisfies:

$$\mathbb{P}_{\mathcal{D}^m} \left[\max_{j \in [m]} \|A_j\|_F \leq R, \text{ \& } \exists X \in \Omega_\varepsilon \text{ s.t. } \|\mathcal{A}(X)\|_2 \leq \delta \right]$$

$$\leq \sum_{i=1}^{N_{\Omega_\varepsilon}(\rho)} \mathbb{P}_{\mathcal{D}^m} \left[\max_{j \in [m]} \|A_j\|_F \leq R, \text{ \& } \exists X \in (X'_i + 2\rho\mathcal{B}_{n_1 \times n_2}), \right.$$

$$\left. \text{s.t. } \|\mathcal{A}(X)\|_2 \leq \delta \right] \quad (4)$$

$$\leq \sum_{i=1}^{N_{\Omega_\varepsilon}(\rho)} \mathbb{P}_{\mathcal{D}^m} \left[\max_{j \in [m]} \|A_j\|_F \leq R, \text{ \& } \exists X \in (X'_i + 2\rho\mathcal{B}_{n_1 \times n_2}), \right.$$

$$\left. \text{s.t. } |\langle A_j, X \rangle| \leq \delta, \forall j \in [m] \right] \quad (5)$$

$$\leq \sum_{i=1}^{N_{\Omega_\varepsilon}(\rho)} \mathbb{P}_{\mathcal{D}^m} \left[\|A_j\|_F \leq R, |\langle A_j, X'_i \rangle| \leq 3\delta, \forall j \in [m] \right] \quad (6)$$

$$= \sum_{i=1}^{N_{\Omega_\varepsilon}(\rho)} \left(\mathbb{P}_{\mathcal{D}} [\|A_1\|_F \leq R, |\langle A_1, X'_i \rangle| \leq 3\delta] \right)^m \quad (7)$$

$$\leq N_{\Omega_\varepsilon}(\rho) \left(C_{\mathcal{D}} \cdot \frac{3\delta}{\varepsilon} \right)^m \quad (8)$$

$$\leq C \left(\frac{\delta}{R} \right)^{-d} \left(C_{\mathcal{D}} \cdot \frac{3\delta}{\varepsilon} \right)^m. \quad (9)$$

Inequality (4) uses a union bound. The event in (4) implies the event in (5), which then implies the event in (6). Inequality (6) is due to the following chain of inequalities, of which the last is implied by $\|A_j\|_F \leq R$, $\|X'_i - X\|_F \leq 2\rho$, and $|\langle A_j, X \rangle| \leq \delta$:

$$|\langle A_j, X'_i \rangle| \leq |\langle A_j, X'_i - X \rangle| + |\langle A_j, X \rangle|$$

$$\leq \|A_j\|_F \|X'_i - X\|_F + |\langle A_j, X \rangle|$$

$$\leq 2R\rho + \delta = 3\delta. \quad (10)$$

Equation (7) is due to the fact that $\{A_j\}_{j=1}^m$ are i.i.d. random matrices. Inequality (8) follows from

$$\mathbb{P}_{\mathcal{D}} [\|A_1\|_F \leq R, |\langle A_1, X'_i \rangle| \leq 3\delta] \leq \mathbb{P}_{\mathcal{D}} [|\langle A_1, X'_i \rangle| \leq 3\delta],$$

and the concentration of measure bound (2). (By construction, X'_i , as points in Ω_ε , satisfy $\|X'_i\|_F > \varepsilon$.) Inequality (9) uses the fact that $N_{\Omega_\varepsilon}(\rho) \leq N_{\Omega_B}(\rho) = N_{\Omega_B}(\frac{\delta}{R})$, and the assumed bound on the covering number. (By assumption, $\frac{\delta}{R} < 1$.) Replacing the first term in (3) by (9), we have

$$P_f \leq C (3C_{\mathcal{D}})^m \cdot R^d \cdot \frac{\delta^{m-d}}{\varepsilon^m} + m \cdot \theta_{\mathcal{D}, R},$$

thus completing the proof. \blacksquare

V. CONCLUSIONS

We studied the optimal sample complexity of the matrix recovery problem. If the constraint set has a small covering number, and the measurement matrices follow distributions that satisfy some concentration of measure inequalities, then under (near) optimal sample complexities, the recovery is stable with high probability against perturbations in the measurements.

REFERENCES

- [1] Y. C. Eldar and G. Kutyniok, *Compressed sensing: theory and applications*. Cambridge University Press, 2012.
- [2] M. A. Davenport and J. Romberg, "An overview of low-rank matrix recovery from incomplete observations," *Preprint*, 2015.
- [3] E. J. Candès and B. Recht, "Exact matrix completion via convex optimization," *Found. Comput. Math.*, vol. 9, no. 6, pp. 717–772, Apr 2009.
- [4] Y. Li, K. Lee, and Y. Bresler, "Identifiability and stability in blind deconvolution under minimal assumptions," *arXiv preprint arXiv:1507.01308*, 2015.
- [5] J. Nzabanita, *Bilinear and Trilinear Regression Models with Structured Covariance Matrices*. Dissertation, Linköping University, 2015.
- [6] E. J. Candès, T. Strohmer, and V. Voroninski, "Phaselift: Exact and stable signal recovery from magnitude measurements via convex programming," *Commun. Pure Appl. Math.*, vol. 66, no. 8, pp. 1241–1274, 2013.
- [7] S. Bahmani and J. Romberg, "Sketching for simultaneously sparse and low-rank covariance matrices," *arXiv preprint arXiv:1510.01670*, 2015.
- [8] E. Candès and T. Tao, "Decoding by linear programming," *IEEE Trans. Inf. Theory*, vol. 51, no. 12, pp. 4203–4215, Dec 2005.
- [9] E. J. Candès, J. K. Romberg, and T. Tao, "Stable signal recovery from incomplete and inaccurate measurements," *Comm. Pure Appl. Math.*, vol. 59, no. 8, pp. 1207–1223, 2006.
- [10] B. Recht, M. Fazel, and P. A. Parrilo, "Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization," *SIAM Review*, vol. 52, no. 3, pp. 471–501, Jan 2010.
- [11] E. Candès and Y. Plan, "Tight oracle inequalities for low-rank matrix recovery from a minimal number of noisy random measurements," *IEEE Trans. Inf. Theory*, vol. 57, no. 4, pp. 2342–2359, April 2011.
- [12] V. Chandrasekaran, B. Recht, P. A. Parrilo, and A. S. Willsky, "The convex geometry of linear inverse problems," *Found. Comput. Math.*, vol. 12, no. 6, pp. 805–849, Oct 2012.
- [13] K. Lee, Y. Wu, and Y. Bresler, "Near optimal compressed sensing of sparse rank-one matrices via sparse power factorization," *arXiv preprint arXiv:1312.0525*, 2013.
- [14] T. T. Cai and A. Zhang, "ROP: Matrix recovery via rank-one projections," *Ann. Stat.*, vol. 43, no. 1, pp. 102–138, Feb 2015.
- [15] A. Ahmed, B. Recht, and J. Romberg, "Blind deconvolution using convex programming," *IEEE Trans. Inf. Theory*, vol. 60, no. 3, pp. 1711–1732, Mar 2014.
- [16] K. Lee, Y. Li, M. Junge, and Y. Bresler, "Blind recovery of sparse signals from subsampled convolution," *arXiv preprint arXiv:1511.06149*, 2015.
- [17] Y. Wu and S. Verdu, "Optimal phase transitions in compressed sensing," *IEEE Trans. Inf. Theory*, vol. 58, no. 10, pp. 6241–6263, Oct 2012.
- [18] G. Reeves, "The fundamental limits of stable recovery in compressed sensing," in *IEEE Int. Symp. Info. Theory (ISIT 2014)*, June 2014, pp. 3017–3021.
- [19] E. Riegler, D. Stotz, and H. Bölcskei, "Information-theoretic limits of matrix completion," *arXiv preprint arXiv:1504.04970*, 2015.